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Marat Ibragimov, Rustam Ibragimov, Jovlon Karimov and Galiya Yuldasheva

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Robust Analysis of Income Inequality Dynamics in Russia:
***t*-Statistic Based Approaches**

Marat Ibragimov[1], Rustam Ibragimov[2], Jovlon Karimov[1], Galiya Yuldasheva[3] [4]

Abstract

Empirical analyses on inequality measurement and those in other fields in economics and finance often face the difficulty that the data is correlated, heterogeneous or heavy-tailed in some unknown fashion. The paper focuses on analogues and modifications of the recently developed *t*-statistic based robust inference methods that are applicable in the analysis of income and wealth distributions and inequality measures. The methods can be used under general conditions appropriate for real-world markets and have several advantages over other inference approaches available in the literature. We illustrate the use of the robust inference approaches in the study of important problems with pronounced complications for alternative econometric procedures focusing on the analysis of income distribution and inequality in the Russian economy where heterogeneity, outliers and crisis effects are expected to be present.

Among other results, the paper provides robust confidence intervals for the Gini coefficient in Russia in the periods before and after the beginning of the on-going crisis. The results considerably complement the point estimates of the Gini coefficient for the Russian economy available in the literature. They further point out to significant changes in income inequality and redistribution of income in Russia prompted by the beginning of the on-going crisis in 2008.

In addition to the above results, we also present characterizations of the whole income distribution in Russia using double Pareto models recently introduced to the field. The empirical results for double power-law models for Russian income distribution point out to its significant heavy-tailedness and provide further motivation for the development and applications of robust approaches to inference on income distributions, inequality measures and their dynamics and structural changes, both in emerging and transition economies and developed markets.

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Keywords: Income inequality, inequality measures, robustness, heavy-tailedness, Russian economy

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[1] Tashkent State University of Economics
[2] Imperial College Business School; Corresponding author, irustam@imperial.ac.uk
[3] Orenburg State University
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1. Introduction

1.1. Income inequality measures and robustness

The main goal of the paper is two-fold. First, it focuses on analogues and modifications of the recently developed robust inference methods that are applicable in the analysis of income and wealth distributions and inequality measures. The methods have a number of appealing properties and advantages over other approaches available in the literature and can be used under general conditions appropriate for real-world markets. Second, it illustrates the use of the robust inference approaches in the study of important problems with pronounced complications for alternative econometric procedures focusing on the analysis of income distribution and inequality in the Russian economy where heterogeneity, outliers and crisis effects are expected to be present. The development and applications of the robust inference approaches in the context of the analysis of income distributions and inequality in the paper overcome several methodological problems. In addition, the estimates and comparisons of inequality and income distribution parameters for different time periods (e.g., before and after the beginning of the on-going crisis) in the paper provide a robust assessment of their structural changes due to crises and other external shocks. Among other results, the paper provides a robust analysis of statistical significance of the obtained estimates of inequality measures and income distribution parameters in the Russian economy and their changes due to the current economic and financial crisis. The empirical analysis qualitatively differs from the previous studies on the topic both in terms of the focus on the on-going crisis and the use of recently developed robust inference methods.

Income and wealth inequality was the focus of numerous studies in economics starting with its formation as a scientific discipline (see, among others, the reviews in Milanovic, 2005, Marshall, Olkin and Arnold, 2011, and Milanovic, 2011). Among others, Milanovic (2005, 2011) provides a comprehensive review of the dynamics of inequality over time in different countries of the World.

Empirical analyses on inequality measurement and those in other fields in economics and finance often face the difficulty that the data is correlated, heterogeneous or heavy-tailed in some unknown fashion. For instance, as has been documented in numerous studies, observations on many variables of interest, including income, wealth and financial returns, typically exhibit heterogeneity, dependence and heavy tails in the form of commonly observed Pareto or power laws (see, among others, the discussion and reviews in Section 1.2 of the paper, Embrechts, Klüppelberg & Mikosch, 1997, Mandelbrot, 1997, Atkinson, 2008, Gabaix, 2008, 2009, Ibragimov, 2009, Milanovic, 2011, and references therein).

Emerging and transition economic and financial markets are more volatile than their developed counter-parts and are subject to more extreme external and internal shocks. Heterogeneity and heavy-tailedness properties are usually even more pronounced for income and wealth distributions, exchange rates and other important economic and financial variables in these markets (see, for instance, Ibragimov, Ibragimov & Kattuman, 2011, Ibragimov, Ibragimov & Khamidov, 2011). In addition, heterogeneity and outliers in observations are typical for the periods of crises and other external shocks.

Applicability of many commonly used inequality measures becomes problematic under heterogeneity, heavy-tailedness and correlation in the data generating process. Several recent works in the literature, for instance, have emphasized robustness as an important aspect in the choice of measures used in assessing economic inequality and poverty and estimation and inference methods for them (see, among others, Cowell & Flachaire, 2007, Davidson & Flachaire, 2007, Zandvakili, 2008, and references therein). In particular, Cowell & Flachaire (2007) and Davidson & Flachaire (2007) advocate the use of computationally expensive alternatives to asymptotic inference methods on income inequality that are based on different parametric and semiparametric bootstrap procedures.

The standard approaches to inequality inference based on asymptotic normality of empirical income inequality measures typically have poor finite sample properties, in part due to outliers and heterogeneous observations generated by heavy-tailed income and wealth distributions. These problems are especially pronounced in periods with structural changes, crises and other external economic shocks. They are also typical for transition and emerging economies with volatile economic indicators and data availability and quality problems. Many studies on the analysis of income inequality across countries and over time often provide only point estimates of inequality measures. Their conclusions may need to be modified following an assessment of standard errors and statistical significance of the reported empirical results.

Recently, Ibragimov & Müller (2010) developed a simple general approach to robust inference about a scalar parameter of interest when the data is potentially heterogeneous and correlated in a largely unknown way. Following the approach, one conducts robust large sample inference as follows: partition the data into $q \geq 2$ groups, estimate the model for each group, and conduct a standard t-test with the resulting q parameter estimators of interest. This results in valid and in some sense efficient inference when the groups are chosen in a way that ensures the parameter estimators to be asymptotically independent, unbiased and Gaussian of possibly different variances. Ibragimov & Müller (2010) provide a number of examples of how to apply this approach to time series, panel, clustered and spatially correlated data.

The data on income distributions in transition and emerging economies affected by the on-going crisis provide natural areas for applications of robust inference approaches due to typical problems of heterogeneity, heavy tails, small sample size and dependence. The present study develops new analogues and modifications of the robust t-statistic based inference approaches that can be used in the analysis of income distributions and inequality under the above complications. It further provides the first applications of the approaches in inequality measurement and related areas, focusing on the important case of the Russian economy affected by the on-going crisis and other external shocks.

As discussed in Ibragimov & Müller (2010), in applications of the t-statistic inference approach, the asymptotic Gaussianity of group estimators of a certain parameter of interest (e.g., an income inequality measure for a country), typically follows from the same reasoning as the asymptotic Gaussianity of the full sample estimator (the empirical inequality measure calculated for the full sample of income observations). The argument for the asymptotic independence of the group estimators, on the other hand, depends on the choice of groups and the details of the application. In the context of inference on income inequality, the independence condition presents methodological problems for direct applications of the t-statistic based approach. The straightforward group choices such as grouping of income by geographical regions are likely to produce asymptotically correlated estimators of income inequality due to (spatial and common shock) dependence among the observations in different groups. The present study proposes a solution to these problems based on randomization of the initial samples of income observations. In the randomization stage, each of the observations is randomly assigned to one of the q groups $j=1, \ldots, q$ with equal probability $1/q$. The t-statistic based approach is then applied to the q groups of consecutive observations in the randomized sample of incomes. That is, the observation i in the randomized sample is an element of group j if $(j-1)N/q < i \leq jN/q$, where N is the sample size of observations. The group estimators of income inequality computed for the randomized sample are independent by construction, and both the conditions for validity of the t-statistic based inference using the group estimators are thus satisfied. The randomization-based modifications of the t-statistic robust inference approach proposed in the paper may also prove to be useful in other problems with inherent dependence among the group estimators dealt with.

Among other results, the results obtained in the paper shed light on changes in the income distribution and inequality in the Russian economy due to the on-going crisis and other external shocks. Naturally, crises and other shocks to an economy often lead to structural changes in key economic indicators, including income and wealth distributions and inequality measures. In turn, as discussed in many studies (see, among others, the reviews in Ibragimov & Ibragimov, 2007, Quadrini 2008, Ibragimov, Ibragimov & Kattuman, 2011, and references therein), the shifts in income and wealth inequality greatly affect economic growth, consumer demand, market equilibrium and different economic variables, thus contributing to their changes due to the crisis.

1.2. Extreme observations, outliers, heavy-tailedness and heterogeneity of income and wealth distributions

As discussed above, an important motivation for the use of robust inference methods is provided by the presence of extreme observations and outliers in the data on important economic and financial variables, including income and wealth. In turn, the analysis of the likelihood of occurrence of outliers and extreme observations in data on income and wealth is directly related to the study of the behavior of distributions of these variables in the upper tails and to estimation of their tail indices.

In models involving a heavy-tailed random variable (r.v.) $Y>0$ it is usually assumed that the distribution of Y has power tails:

$$P(Y > y) \sim Cy^{-\zeta} \tag{1}$$

as $y \to \infty$, with the tail index $\zeta > 0$ (here and throughout the paper, $f(y) \sim g(y)$ means that $f(y) = g(y)(1 + o(1))$ as $y \to \infty$). The tail index ζ characterizes the heaviness (the rate of decay) of the tails of power law distribution (1). An important property of r.v.'s Y satisfying a power law with the tail index ζ is that the moments of Y are finite if and only if their order is less than ζ: $E(Y^p) < \infty$ if and only if $p < \zeta$. Heavy-tailedness (the tail index ζ) of the variable Y (e.g., income, wealth, risk, financial return or foreign exchange rate) governs the likelihood of observing outliers and large fluctuations in the variable. The smaller values of the tail index ζ correspond to a higher degree of heavy-tailedness in Y and, thus, to a larger likelihood of observing outliers and large fluctuations in the realizations of this variable. Important examples of power laws are given by Pareto (C8) and double Pareto (D1)-(D3) distributions discussed in Appendices C and D where (1) holds as equality for all y's greater than some threshold.

Empirical studies of income and wealth indicate that distributions of these variables in developed economies typically satisfy power laws (1) with the tail index ζ that varies between 1.5 and 3 for income and is rather stable, perhaps around 1.5, for wealth (see, among others, Atkinson and Piketty, 2007, Atkinson, Piketty and Saez, 2011, Gabaix, 2008, 2009, and references therein). This implies, in particular, that the mean is finite for income and wealth distributions (since $\zeta > 1$). However, the variance is infinite for wealth (since $\zeta \approx 1.5 < 2$) and may be infinite for income (if $\zeta \leq 2$). In addition, since their tail indices are smaller than 3, income and wealth distributions have infinite third and higher moments. The problem of infinite variance in income and wealth distributions is important because, as indicated above, it may invalidate or make problematic direct applicability of standard inference approaches, including regression analysis and least squares methods. In a similar fashion, infinite fourth moments for these variables need to be taken into account in regression and other models involving their volatilities or variances, e. g., in the analysis of permanent and transitory components of income variability and their cross-country comparisons (see Gorodnichenko, Peter and Stolyarov, 2010) and the study of autocorrelation properties of financial time series (see Cont, 2001, and references therein). Finiteness of first moments is important because it points out to optimality of diversification and robustness of a number of economic models for the variables considered (see Ibragimov, 2009, and Ibragimov, Jaffee and Walden, 2009).[5]

In addition to the tail behavior analysis, it is also of great importance to characterize the whole distribution of income or wealth, including its behavior in the left tails (poor population) and for the middle range of incomes and wealth levels, in part since income and wealth distributions in different parts of their support affect the overall inequality, economic, economic conditions and social stability.

In this direction, in addition to the robust analysis of income distribution and inequality in Russia, the paper focuses on characterization of the whole income distribution in the country using double power laws recently introduced to the field by Toda (2012). The empirical results for double Pareto models for Russian income distribution point out to its significant heavy-tailedness. These results further motivate the development and use of robust approaches to inference on income distributions, inequality measures and their dynamics and structural changes, both in emerging and transition economies and developed markets.

1.3. Organization of the paper

The paper is organized as follows. Section 2 reviews the data used in the study. Section 3 describes the new analogues and modifications of the *t*-statistic based robust inference approaches used in the empirical analysis in the paper. Section 4 presents and discusses the main empirical results obtained. Section 5 makes some concluding remarks. Appendix A provides tables and diagrams on the estimation results. Appendix B discusses the definitions and basic properties of several widely used income inequality measures that are considered in the paper. Appendix C.1 reviews the general *t*-statistic based approach to robust inference under heterogeneity, correlation and heavy tails in Ibragimov & Müller (2010). Appendix C.2 provides the numerical results on finite sample performance of the approach in inference on widely used measures of income inequality comparing to alternative methods. We also present, in Appendix C.3, new theoretical results that indicate the

[5] Many recent studies argue, using the data for developed economies, that the tail indices ζ typically lie in the interval (2, 4) for many financial returns and exchange rates. For instance, among other results, Gabaix, Gopikrishnan, Plerou & Stanley (2006) present and discuss empirical estimates that support heavy-tailed distributions with tail indices $\zeta \approx 3$ for financial returns on many stocks and stock indices in developed markets. These results imply that, in contrast to income and wealth distribution, financial returns have finite variance (since $\zeta > 2$). Similar to the case of income and wealth distributions, financial returns have infinite fourth moments ($\zeta < 4$) and may have infinite moments of order 3 (if $\zeta \leq 3$).

connection of the small sample conservativeness properties of *t*-statistics used in the robust inference approaches to income inequality measures and other important economic indicators. Appendix D reviews the definition and properties of double Pareto distributions used in modeling double power law behavior of income in Russia.

2. Data

The study of income distribution and inequality dynamics in Russia in this paper is based on quarterly household budget survey results from the beginning of 2003 to the end of 2009 in this country provided by the Rosstat (the Federal State Statistics Service of Russian Federation, http://www.micro-data.ru). The household survey results provided by the Rosstat contain microdata on a number of variables, including household income, disposable resources, consumption and detailed expenditures. The dataset thus contains the total of 28 quarterly observations on various household budget indicators for 45,000-53,000 households in Russia (see Table 1 for household sample sizes in the dataset).

The analysis in the paper focuses on household total incomes (disposable resources) that, in addition to money resources of households, also include their natural receipts of foodstuffs and in-kind subsidies and benefits. Thus, comparing to the levels of households' monetary income and disposable income, the volumes of their total income better reflect the households' real opportunities for personal consumption and savings.

Table 1 in Appendix A presents the main descriptive statistics for quarterly data on total income of Russian households in 2003-2009 (see the data on the above Microdata Household Budget Surveys in Russia at http://www.micro-data.ru). Table 2 and Figure 1 in the appendix provide the quarterly Gini coefficients calculated for distribution of income among Russian households. To illustrate the degree of inequality in the lower and upper tails of Russian income distribution, we also present, in Table 2 and Figure 1, the Gini coefficients for household income levels that are less than the modal value and for those that are greater than the mode.

3. Methodology: Randomization-based analogues and modifications of *t*-statistic based robust inference approaches for income distributions and inequality

One of the main focuses of the empirical analysis in the paper is on applications of the new *t*-statistic based correlation and heterogeneity robust inference approaches recently developed in Ibragimov & Müller (2010). These approaches are reviewed in detail, in the general case of inference about an arbitrary scalar parameter, in Appendix C.1.

For an illustration of the t-statistic based approach to robust income inequality measurement, consider the problem of statistical inference on an income inequality measure $\mathcal{L}$ (for instance, the Gini coefficient or Theil index, see the review and discussion in Appendix B). As is common in income inequality analysis, let the data generating processes exhibit heavy tails, heterogeneity or dependence. The t-statistic based robust test of level $\alpha < 5\%$ of the hypothesis H_0: $\mathcal{L}=\mathcal{L}_0$ against the alternative H_a: $\mathcal{L} \neq \mathcal{L}_0$ is performed as follows: partition the sample $I_1, I_2,..., I_N$ of observations on incomes into $q\geq 2$ groups, estimate the income inequality measure $\mathcal{L}$ for each group thus obtaining the group empirical income inequality measures $\hat{\mathcal{L}}_j, j = 1,\ldots,q,$ and reject H_0 in favor of H_a when $|t_{\mathcal{L}}|$ exceeds the $(1-\alpha/2)$-percentile t_α of the Student-t distribution with $q-1$ degrees of freedom, where $t_{\mathcal{L}}$ is the usual t-statistic $t_{\mathcal{L}} = \sqrt{q}\frac{\bar{\hat{\mathcal{L}}}-\mathcal{L}_0}{s_{\hat{\mathcal{L}}}}$ with $\bar{\hat{\mathcal{L}}} = \frac{\sum_{j=1}^{q}\hat{\mathcal{L}}_j}{q}$ and $s_{\hat{\mathcal{L}}}^2 = \frac{\sum_{j=1}^{q}\left(\hat{\mathcal{L}}_j-\bar{\hat{\mathcal{L}}}\right)^2}{q-1}$ (the $(1-\alpha/2)$-quantile t_α satisfies $P(|T_{q-1}| > t_\alpha)= \alpha$, where T_{q-1} is a random variable that has the Student-t distribution with $q-1$ degrees of freedom). As follows from Ibragimov & Müller (2010), the above procedure results in asymptotically valid and in some sense efficient inference when the groups are chosen in a way that ensures the group empirical income inequality measures $\hat{\mathcal{L}}_j$, $j=1,\ldots,q,$ to be asymptotically independent, unbiased and Gaussian of possibly different variances. Furthermore, the asymptotic validity of the t-statistic based inference approach continues to hold even when the group estimators $\hat{\mathcal{L}}_j$ of $\mathcal{L}$ converge (at an arbitrary rate) to independent but potentially heterogeneous mixed normal distributions, such as the family of stable symmetric distributions, or to conditionally normal variates which are unconditionally dependent through their second moments. In particular, the t-statistic based robust inference on $\mathcal{L}$ can thus be applied under heavy tails, extremes and outliers in observations and, among others, dependence structures that include models with multiplicative common shocks and their convolutions (see Ibragimov, 2009).

As discussed in Ibragimov & Müller (2010), in applications of the t-statistic inference approach, the asymptotic Gaussianity of group estimators $\hat{\beta}_j$, $j=1,\ldots,q,$ of a certain parameter of interest β (e.g., income inequality measure $\mathcal{L}$ for a country), typically follows from the same reasoning as the asymptotic Gaussianity of the full sample estimator $\hat{\beta}^N$ (the empirical inequality measure $\mathcal{L}^N$ computed for the full sample $I_1, I_2,..., I_N$ of income observations). The argument for the asymptotic independence of the group estimators $\hat{\beta}_i$ and $\hat{\beta}_j$ for $i=j$, on the other hand, depends on the choice of groups and the details of the application. In the context of inference on income inequality, the asymptotic independence condition presents methodological problems for direct applications of the t-statistic based approach. The straightforward group choices such as grouping of income by geographical regions are likely to produce asymptotically correlated estimators $\hat{\mathcal{L}}_j$ of income inequality due to (spatial and common shock) dependence among the observations in different groups. The present study proposes a solution to these problems based on randomization of the initial samples $I_1, I_2, \ldots, I_N$ of income observations. In the randomization stage, each of the observations I_i, $i = 1, \ldots, N$, is randomly assigned to one of the q groups $j = 1, \ldots, q$ with equal probability $1/q$. The t-statistic based approach is then applied to the q groups of consecutive observations in the randomized sample of incomes $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_N$. That is, the observation i in the randomized sample is element of group j if $(j-1)N/q < i \leq jN/q$. The group estimators of income inequality calculated for the randomized sample are independent by construction, and both the conditions for validity of the t-statistic based inference using the estimators are thus satisfied. The randomization-based modification of the t-statistic robust inference approach proposed in the paper may also prove to be useful in other problems with inherent dependence among the group estimators dealt with.

It is important to note that, in the standard way, the above results on the t-statistic based robust tests on the measure $\mathcal{L}$ (e.g., the Gini coefficient for Russian income distribution as in the next section) allow one to construct robust confidence intervals for this indicator. In particular, given the q group estimators $\hat{\mathcal{L}}_j$, $j=1,\ldots,q$ (e.g., the group empirical Gini coefficients computed for the randomized sample as described), the $(1-\alpha)$-confidence interval for $\mathcal{L}$ is given by $(\bar{\hat{\mathcal{L}}} - t_\alpha s_{\hat{\mathcal{L}}},\ \bar{\hat{\mathcal{L}}} + t_\alpha s_{\hat{\mathcal{L}}})$, where, as before, t_α denotes the $(1-\alpha/2)$-quantile of the Student-t distribution with $q-1$ degrees of freedom. For instance, the 95% confidence interval for $\mathcal{L}$ is given by $(\bar{\hat{\mathcal{L}}} - t_{0.05} s_{\hat{\mathcal{L}}},\ \bar{\hat{\mathcal{L}}} + t_{0.05} s_{\hat{\mathcal{L}}})$, where $t_{0.05}$ is the 0.975-quantile of the Student-t distribution with $q-1$ degrees of freedom: $P(|T_{q-1}| > t_{0.05})=0.05$.

Robust estimates and confidence intervals for indicators of interest over time periods before and after the beginning of a crisis (or another structural break date) allow one to provide an assessment of their changes due to this shock (see the next section for the analysis of changes in the Gini coefficient for the distribution of income in the context of the on-going crisis).

The numerical results on t-statistic based approach to robust inequality measurement in Appendix C.2 indicate its appealing finite sample properties and applicability to many widely used income inequality measures, including the Gini coefficient; Theil index, mean logarithmic deviation and generalized entropy measures. Appendix C.3 further presents several theoretical results that imply a strong link between the t-statistic based robust inference methods and stochastic analogues of majorization conditions that are usually imposed on inequality measures related to self-normalized sums or their transforms, as in the case of the coefficient of variation.

As discussed in Section 1, in addition to the robust analysis of income inequality in Russia, we further present estimation results for heavy-tailed double Pareto models for income distribution in this country (see Appendix D). These results provide further motivation for applications of the robust inference approaches discussed in this section and throughout the paper.

4. Empirical results

Importantly, the results in Table 1 on descriptive statistics for income distribution among Russian households indicate extremely large values for kurtosis of the distribution that point out to its heavy-tailedness. According to Table 2 on quarterly Gini coefficients for the whole distribution of income among households in Russia and its parts on the left and the right from the mode, in the second half of 2007 and in 2008, the Gini coefficient in Russia has achieved its maximum in the considered period from the beginning of 2003 to the end of 2009. The Gini coefficient has started declining after 2008.The Gini coefficient for the left part of income distribution (among households with income levels less than the mode) has not been significantly affected by the on-going crisis. The Gini coefficient for the right part of income distribution (households with income levels greater than the mode) has increased in 2008. However, the Gini coefficient for this part of the distribution has returned to its pre-crisis level already by the first quarter of 2009.

Table 3 and Figure 2 provide the maximum likelihood estimates of the left and right tail index (shape) parameters k_1 and k_2 for the double Pareto and the truncated double Pareto distributions discussed in Appendix D. As discussed in the appendix, in the case of the truncated double Pareto distribution, the truncation bound b is estimated by the maximal income level in the sample $Y=\{y_1,\dots, y_n\}$ of observations on household incomes: $\hat{b} = \max_{1\le i\le n} y_i$. Importantly, in 2009, one observes a decrease in the degree of heavy-tailedness of household income distribution in Russia comparing to 2008. This can be seen from an increase in the tail index $\zeta=k_2$ in double Pareto model (D1)-(D3) for the income distribution.

Most of the estimates for the right tail index coefficient k_2 (that is, the upper tail index $\zeta = k_2$ in (1)) of the income distribution in Russia in Table 3 are considerably less than the benchmark values $\zeta\in(1.5,\ 3)$ that are typically obtained for the tail indices of income distributions in developed economies (see Section 1.2 in the paper, Gabaix, 2008, 2009, and references therein). They are further considerably smaller than the estimates of the upper tail index k_2 for the double Pareto family fitted to the personal income distribution in the US in 1968-1993 conditional on education and experience in Toda (2012). In particular, most of the estimates of the upper tail index k_2 for the double Pareto family fitted to the Russian income distribution in Table 3 are considerably smaller than the threshold value $\zeta = 2$ implying infinite variances.

The above differences of the estimates of the tail index k_2 for Russian income distribution from the results in Toda (2012) for the US may be due to pre-crisis datasets considered in that paper (the data for 1968-1993) and conditioning, but may also be indicative that heavy-tailedness properties and the likelihood of observing extreme observations and outliers are significantly more pronounced in income distribution in Russia comparing to the US and other developed economies. The latter conclusion joins and complements those for exchange rates in emerging markets in Ibragimov, Ibragimov & Kattuman, 2011, where, qualitatively similar to this paper, the estimates obtained also point out to smaller tail index values and significantly more pronounced heavy-tailedness properties comparing to the tail indices $\zeta \in (2, 4)$ for financial returns and exchange rates in developed markets reported in the literature (see Gabaix, 2008, 2009, Ibragimov, 2009, and references therein). As discussed in Section 1, the observed empirical facts on heavy-tailedness, extreme observations and outliers in key variables in emerging and transition economies like Russia, such as income, exchange rates and financial returns, provide further motivation for applications of robust inference methods, e.g., the *t*-statistic based robust inference approaches like in this paper, in their analysis.

Table 4 and Figures 3 and 4 provide the confidence intervals for the Gini coefficient for income distribution in Russia constructed using the *t*-statistic based robust inference approach with q=4 and q=8 equal-sized groups applied to (randomized) income data as described in Section 3 and Appendix C.1. According to the Table 4, the robust 95% confidence intervals for the Gini coefficient for income distribution in Russia in the 4th quarters of 2003-2009 constructed using the *t*-statistic approach with q=8 are as follows: CI_{2003}=[0.380, 0.390], CI_{2004}=[0.386, 0.397], CI_{2005}=[0.394, 0.409], CI_{2006}=[0.400, 0.413], CI_{2007}=[0.414, 0.431], CI_{2008}=[0.396, 0.421] and CI_{2009}=[0.378, 0.390].

It is important to note that these and other results on confidence intervals for quarterly Gini coefficients in Russia in Table 4 considerably complement the point estimates of Gini coefficients for Russia and other economies available in the literature and are largely in accordance with them. In particular, Milanovic (2005, 2011) discusses the Gini percentage points in the mid-twenties for Nordic European countries that are the most egalitarian, and around 30 to 35% for Western and Central Europe, Australia, Canada, New Zealand and a number of Asian economies (India, Japan and Taiwan). As discussed in Milanovic (2005, 2011), the Gini percentage points are in the lower 40s for the US and Russia, in excess of 40 for China, in the 50s for countries in Latin America and Africa, and are almost

60 for some countries like Brazil, South Africa or Botswana.[6] Table 5 in Appendix A provides the official values of Gini coefficients for income distribution in Russia in 1995-2009 from the Federal State Statistics Service of Russian Federation (Rosstat, see also Section 2). According to the table, the official values of the Gini coefficient for the whole income distribution in Russia increased from about 0.3 in 1992 to relatively stable values about 0.4 in 1995-2009. Gorodnichenko *et al.* (2010) present a detailed analysis of Russian income inequality dynamics using a panel micro data set from the Russia Longitudinal Monitoring Survey (RLMS) for 1994–2005 (see also Kislitsyna, 2003, for a review of the dynamics of income inequality in Russia in the 1990s). The authors argue that, after an initial rise in the early 1990s, the measures of income inequality in Russia stayed relatively high during 1994-1998 and then started falling during 2000-2005. According to the estimates in the paper, the Gini coefficient for earnings in Russia increased from 0.28 in 1985 and 0.32 in 1990 to 0.48 in 1995 and then declined to 0.41 in 2005 (see also Kislitsyna, 2003). As discussed in the paper, the latter value of the Gini coefficient in 2005 is just slightly higher than the mean value of Gini coefficients for after-tax household income in upper middle income countries.

Furthermore, the analysis of the confidence intervals points out to several important conclusions on the dynamics of the Gini coefficient in Russia and the effects of the on-going crisis on income distribution and inequality in that country. Importantly, for the number of groups *q*=8, the confidence interval CI_{2007}=[0.414, 0.431] for the Gini coefficient in Russia in the 4th quarter of 2007 does not intersect with the confidence intervals CI_{2003}=[0.380, 0.390], CI_{2004}=[0.386, 0.397], CI_{2005}=[0.394, 0.409], CI_{2006}=[0.400, 0.413] and CI_{2009}=[0.378, 0.390]: $CI_i \cap CI_{2007}$ for *i*=2003, ..., 2006 and i=2009. More precisely, the confidence interval CI_{2007} for the Gini coefficient in the 4th quarter of 2007 lies on the right of the latter confidence intervals CI_i for *i*=2003, ..., 2006 and i=2009. This implies that the difference between the Gini coefficient in Russia at the end of 2007 and the Gini coefficients in that country at the end of 2003, 2004, 2005 and 2006 and, most importantly, at the end

[6] See also the World Bank's World Development Indicators (WDI) database (http://databank.worldbank.org) for Gini coefficients in different countries of the World from 2002 to 2010. The All the Ginis database (http://econ.worldbank.org/projects/inequality) provides a compilation of Gini coefficients for different countries and time periods from five databases, including the World Institute for Development Research WIDER dataset (see http://www.wider.unu.edu/wiid/wiid.htm and links to other datasets on income distributions and Gini coefficients therein) and the World Income Distribution (WYD) dataset used in Milanovic (2005). The website at http://econ.worldbank.org/projects/inequality also contains the Global Income Inequality database with the aggregate data on household surveys in the countries of the World in approximate 5 year intervals from 1988 to 2005 used in Milanovic (2005, 2011) (see also the Decomposing World Income Distribution and Globalization and Income Distribution datasets available at the website). The Online Household Expenditure and Income Data for Transitional Economies (HEIDE) database available at the latter website contains the (disaggregated) data on household expenditure and income from surveys conducted in five post-Soviet countries (Russia, 1993-94; Armenia, 1994; Estonia, 1995; Kyrgyz Republic, 1993; and Latvia, 1997-98) and four countries in Eastern Europe (Bulgaria, 1995; Hungary, Poland and Slovak Republic, 1993). Related data on the results of household surveys in different countries and time periods is available from the Living Standards Measurement Study at the World Bank website at http://econ.worldbank.org/projects/inequality . The website at https://www.cpc.unc.edu/projects/rlms-hse contains the results of the Russia Longitudinal Monitoring Survey – Higher School of Economics (RLMS-HSE, see also Gorodnichenko *et al.,* 2010). Among a number of other variables, the RLMS-HSE project provides the data on household income, assets and expenditures in Russia collected in 18 rounds of the study from 1992 to 2009.

of 2009 is statistically significant (at the 5% significance level). More precisely, the Gini coefficient for income distribution in Russia at the end of 2007 was statistically significantly greater (at the 2.5% level) than the Gini coefficients for the income distribution at the end of 2003, 2004, 2005, 2006 and at the end of 2009, the year after the beginning of the on-going crisis in 2008. This conclusion suggests that, apparently, there were significant changes in income distribution and redistribution of income in Russia prompted by the beginning of the on-going crisis in 2008.[7]

At the same time, in contrast to the above, the confidence intervals CI_{2003}, CI_{2004}, CI_{2005}, CI_{2006} and CI_{2009} for the Gini coefficient in Russia in the 4th quarters of 2003-2006 and 2009 do intersect. This means that the Gini coefficients for income distribution among Russian households at the end of 2003-2006 and 2009 are not statistically different from each other (at the 5% significance level). Similar comparisons of robust confidence intervals in Table 4 and conclusions on statistical significance or its lack for differences between Gini coefficients in Russia can be obtained for other time periods before and after the beginning of the current crisis. Furthermore, in a similar fashion, robust confidence intervals can be constructed for other important economic indicators, with applications to the analysis of their structural changes over time and due to the on-going crisis.

5. Conclusion and further research

Empirical analyses on inequality measurement and those in other fields in economics and finance often face the difficulty that the data is correlated, heterogeneous or heavy-tailed in some unknown fashion. In particular, as has been documented in numerous studies, observations on many variables of interest, including income, wealth and financial returns, typically exhibit heterogeneity, dependence and heavy tails in the form of commonly observed Pareto or power laws.

Emerging and transition economic and financial markets are more volatile than their developed counter-parts and are subject to more extreme external and internal shocks. Heterogeneity and heavy-tailedness properties are usually even more pronounced for income and wealth distributions, exchange rates and other key economic and financial variables in these markets, especially in the

[7] Somewhat qualitatively similar conclusions are also obtained for the dynamics and the upper tail inequality of wealth distribution in Russia. Unreported results on estimation of the tail index ζ in (1) for wealth distribution among Russian billionaires using the modifications of log-log rank-size regressions with optimal shifts in ranks and correct standard errors developed in Gabaix and Ibragimov (2011) (see Ibragimov, Ibragimov and Khamidov, 2011, for applications of the regression approaches to estimation of tail indices for the World wealth distribution) provide the following confidence intervals for the tail index in 2008-2011: $\zeta_{2008} \in (0.67, 1.24)$, $\zeta_{2009} \in (0.71, 2.08)$, $\zeta_{2010} \in (0.68, 1.42)$, $\zeta_{2011} \in (0.75, 1.31)$. The comparison of the confidence intervals indicates some increase in the tail index ζ for the wealth distribution in Russia following the beginning of the on-going crisis in 2008, implying a decrease in the degree of heavy-tailedness of the distribution and the corresponding decrease in the upper tail wealth inequality (see Atkinson, 2008, Atkinson, Piketty and Saez, 2011, and Ibragimov, Ibragimov and Khamidov, 2011, for a discussion of the relation of the tail index ζ to income and wealth concentration and inequality among the rich).

periods of crises and other external shocks.

The paper focuses on randomization-based analogues and modifications of t-statistic based robust inference methods recently developed in Ibragimov & Müller (2010) that are applicable in the analysis of income and wealth distributions and inequality measures. The methods can be used under general conditions appropriate for real-world markets and have a number of appealing properties and advantages over other inference approaches available in the literature. We illustrate the use of the robust inference approaches in the study of important problems with pronounced complications for alternative econometric procedures focusing on the analysis of income distribution and inequality in the Russian economy where heterogeneity, outliers and crisis effects are expected to be present. The development and applications of the t-statistic based robust inference approaches in the context of the analysis of income distributions and inequality in the paper overcome several methodological problems related to the condition of asymptotic independence of group empirical inequality measures.

Among other results, the paper provides robust confidence intervals for the Gini coefficients for income distribution in Russia in the periods before and after the beginning of the on-going crisis. The results considerably complement the point estimates of the Gini coefficients for Russian and other economies available in the literature. They further point out to statistically significant changes in income inequality and redistribution of income in Russia prompted by the beginning of the on-going crisis in 2008.

In addition to the above results, we also present characterizations of the whole income distribution in Russia using double Pareto models. The empirical results for double power-law models for Russian income distribution point out to its significant heavy-tailedness and provide further motivation for the development and applications of robust approaches to inference on income distributions, inequality measures and their dynamics and structural changes, both in emerging and transition economies and developed markets.

As follows from the on-going work in progress by R. Ibragimov and U. K. Müller, conservativeness properties similar to those for t-statistics also hold for Behrens-Fisher statistics for testing equality of means: that is, for commonly used significance levels, the Behrens-Fisher tests remain conservative for underlying observations that are independent and Gaussian with heterogenous variances. These small sample conservativeness results provide the basis for the development of asymptotic robust inference procedures using Behrens-Fisher statistics in the latter work and their applications in a number important problems including tests for structural breaks and the analysis of treatment effects. Similar to the t-stastistic based approach, the large sample inference in the Behrens-Fisher case, e.g., robust tests for changes in a parameter of interest (for instance, an income inequality measure), can be conducted as follows: partition the data into some number of groups, estimate the parameter for each group (e.g., calculate the group empirical income inequality measures), and then conduct the standard Behrens-Fisher test on equality of parameters (the no-change hypothesis for the inequality measure or the no-break hypothesis of equality of pre- and post-break parameters) with the resulting group estimators. Applications of the Behrens-Fisher statistic based approaches and their analogues to robust tests for structural breaks in inequality measures and the parameters of income and wealth distributions, together with other applications of the robust inference methods based on conservativeness of test statistics, are currently under way by the authors.

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APPENDICES

A. Tables and figures on estimation results

Table 1. Income distribution among Russian households: Descriptive statistics

Quarter	Sample size	Mode	Median	Mean	Standard deviation	Max	Skewness	Kurtosis
2003,1	53149	8480	14730	18447	15839	892095	8.5	260.6
2003,2	46364	10600	14770	18977	18724	967437	12.1	396.7
2003,3	46364	8700	15827	20534	19835	629594	7.9	146.6
2003,4	46364	8700	17627	22622	21309	790397	8.3	170.9
2004,1	46364	12800	17154	21989	22244	1536014	18.9	902.6
2004,2	46364	10500	17642	22626	22924	1355377	16.0	669.1
2004,3	46364	11200	18726	24430	24762	1315907	11.3	329.1
2004,4	46341	13400	20640	26889	28773	1882243	17.5	748.5
2005,1	46974	12800	20405	26299	26607	1433427	13.3	429.7
2005,2	53132	11100	21860	28603	32241	1890178	17.1	668.4
2005,3	53129	13800	23182	30911	34780	1916806	14.2	480.1
2005,4	53135	16200	25893	34167	35610	1354797	9.9	212.6
2006,1	53093	18200	25705	33271	36992	2156332	18.1	731.8
2006,2	53094	16700	26302	34806	42660	2908380	20.0	854.4
2006,3	53089	15100	27817	37340	43694	2012054	13.9	402.6
2006,4	53072	16600	30969	41122	50145	3781912	24.8	1403.1
2007,1	50589	20400	30567	40094	47450	2302600	15.6	466.9
2007,2	49884	17100	32095	43063	53991	2900069	19.2	741.3
2007,3	53104	19500	35259	48342	66240	4389955	20.7	842.4
2007,4	53096	19100	40335	54716	78176	4707872	20.4	734.7
2008,1	51288	27500	41076	54064	67347	3477145	18.4	632.6
2008,2	51296	20000	43470	58714	79701	3388236	16.6	475.4
2008,3	51292	19800	47478	64568	88027	6542704	19.7	855.6
2008,4	51300	32200	51853	68868	131174	21001918	90.8	13172.4
2009,1	51285	32400	48876	61993	68908	5297012	22.2	1120.4
2009,2	45094	32600	50119	63934	64592	4755511	17.4	872.2
2009,3	51300	26700	52577	67864	66677	3163979	9.4	219.3
2009,4	51309	37200	55569	70907	73460	3143475	11.1	248.9

Table 2. Gini coefficients for income distribution among Russian households (Column 2: all households; Column 3: household income levels less than the modal value; Column 4: household income levels greater than the modal value)

Quarter	Income	Income<mode	Income >mode
1	2	3	4
2003,1	0.37	0.15	0.31
2003,2	0.38	0.17	0.30
2003,3	0.39	0.15	0.33
2003,4	0.39	0.14	0.34
2004,1	0.38	0.17	0.29
2004,2	0.38	0.15	0.32
2004,3	0.39	0.16	0.33
2004,4	0.39	0.16	0.32
2005,1	0.38	0.16	0.31
2005,2	0.40	0.14	0.35
2005,3	0.41	0.16	0.34
2005,4	0.40	0.17	0.33
2006,1	0.39	0.17	0.31
2006,2	0.40	0.16	0.33
2006,3	0.41	0.16	0.35
2006,4	0.41	0.15	0.35
2007,1	0.40	0.17	0.32
2007,2	0.41	0.15	0.35
2007,3	0.43	0.16	0.36
2007,4	0.42	0.14	0.37
2008,1	0.40	0.17	0.33
2008,2	0.42	0.14	0.37
2008,3	0.42	0.14	0.38
2008,4	0.41	0.17	0.34
2009,1	0.38	0.16	0.30
2009,2	0.38	0.16	0.31
2009,3	0.39	0.15	0.34
2009,4	0.38	0.17	0.31

Table 3. Maximum likelihood estimates of the parameters of double Pareto and truncated double Pareto distributions (Appendix D) for income distribution among Russian households

Quarter	n	n_1	Mode	Parameters of double Pareto distribution		Truncation bound, b	Parameters of truncated double Pareto distribution	
				k_1	k_2		k_1	k_2
Q1,2003	53149	10787	8480	3.4307	1.1895	892095	3.4357	1.1888
Q2,2003	46364	14352	10600	2.7309	1.8796	967437	2.7310	1.8796
Q3,2003	46364	8889	8700	3.4249	1.1179	629594	3.4356	1.1166
Q4,2003	46364	7380	8700	3.8122	1.0366	790397	3.8262	1.0354
Q1,2004	46364	15336	12800	2.4653	1.4251	1536014	2.4661	1.4248
Q2,2004	46364	10631	10500	3.1066	1.1929	1355377	3.1100	1.1924
Q3,2004	46364	10784	11200	3.0320	1.1804	1315907	3.0359	1.1797
Q4,2004	46341	12478	13400	2.7580	1.2541	1882243	2.7599	1.2537
Q1,2005	46974	11841	12800	2.9221	1.2348	1433427	2.9251	1.2342
Q2,2005	53132	8998	11100	3.6847	1.0520	1890178	3.6912	1.0514
Q3,2005	53129	12620	13800	2.8804	1.1504	1916806	2.8840	1.1498
Q4,2005	53135	13867	16200	2.7051	1.1930	1354797	2.7099	1.1919
Q1,2006	53093	16520	18200	2.4990	1.3477	2156332	2.5003	1.3473
Q2,2006	53094	14118	16700	2.7311	1.2197	2908380	2.7328	1.2194
Q3,2006	53089	11094	15100	3.0718	1.0688	2012054	3.0779	1.0680
Q4,2006	53072	10680	16600	3.1727	1.0665	3781912	3.1763	1.0660
Q1,2007	50589	14660	20400	2.5770	1.2707	2302600	2.5791	1.2702
Q2,2007	49884	10025	17100	3.2125	1.0624	2900069	3.2177	1.0618
Q3,2007	53104	11682	19500	2.9239	1.0700	4389955	2.9272	1.0696
Q4,2007	53096	8668	19100	3.5902	0.9787	4707872	3.5967	0.9782
Q1,2008	51288	14825	27500	2.6033	1.2788	3477145	2.6050	1.2783
Q2,2008	51296	7496	20000	3.9397	0.9708	3388236	3.9506	0.9701
Q3,2008	51292	6324	19800	4.1975	0.9005	6542704	4.2069	0.9000
Q4,2008	51300	13112	32200	2.7403	1.1967	21001918	2.7407	1.1966
Q1,2009	51285	14022	32400	2.7706	1.3091	5297012	2.8922	1.2113
Q2,2009	45094	12129	32600	2.7657	1.2735	4755511	2.8944	1.5179
Q3,2009	51300	8831	26700	3.5878	1.0498	3163979	3.5971	1.0490
Q4,2009	51309	14476	37200	2.6868	1.3089	3143475	2.6894	1.3081

Note: n is the sample size of households and n_1 is the number of observations on household income levels less than the modal value.

Table 4. Robust *t*-statistic based 95% confidence intervals for the Gini coefficients for income distribution among Russian households in 2003-2009

Quarters	**Sample size, n**	***q*=4**		***q*=8**	
		$\bar{G} = \frac{\sum_{i=1}^{q} \bar{G}_i}{q}$	**95% CI**	$\bar{G} = \frac{\sum_{i=1}^{q} \bar{G}_i}{q}$	**95% CI**
2003:1	53149	0.3668	[0.3603, 0.3734]	0.3668	[0.3629, 0.3707]
2003:2	46364	0.3802	[0.3729, 0.3875]	0.3802	[0.3746, 0.3858]
2003:3	46364	0.3878	[0.3793, 0.3963]	0.3878	[0.3822, 0.3934]
2003:4	**46364**	**0.3852**	**[0.3826, 0.3877]**	**0.3852**	**[0.3804, 0.3899]**
2004:1	46364	0.3790	[0.3696, 0.3884]	0.3790	[0.3743, 0.3838]
2004:2	46364	0.3841	[0.3758, 0.3924]	0.3841	[0.3772, 0.3910]
2004:3	46364	0.3923	[0.3833, 0.4014]	0.3923	[0.3862, 0.3985]
2004:4	**46341**	**0.3915**	**[0.3858, 0.3972]**	**0.3915**	**[0.3858, 0.3971]**
2005:1	46974	0.3847	[0.3699, 0.3994]	0.3846	[0.3766, 0.3926]
2005:2	53132	0.3954	[0.3921, 0.3986]	0.3954	[0.3906, 0.4002]
2005:3	53129	0.4069	[0.4006, 0.4132]	0.4069	[0.4030, 0.4109]
2005:4	**53135**	**0.4019**	**[0.3910, 0.4127]**	**0.4018**	**[0.3942, 0.4093]**
2006:1	53093	0.3905	[0.3843, 0.3968]	0.3905	[0.3850, 0.3960]
2006:2	53094	0.4042	[0.3869, 0.4214]	0.4042	[0.3961, 0.4122]
2006:3	53089	0.4137	[0.4126, 0.4148]	0.4137	[0.4084, 0.4190]
2006:4	**53072**	**0.4065**	**[0.3974, 0.4157]**	**0.4065**	**[0.3997, 0.4133]**
2007:1	50589	0.4010	[0.4004, 0.4017]	0.4010	[0.3959, 0.4061]
2007:2	49884	0.4132	[0.4029, 0.4235]	0.4131	[0.4020, 0.4243]
2007:3	53104	0.4270	[0.4142, 0.4397]	0.4268	[0.4180, 0.4357]
2007:4	**53096**	**0.4226**	**[0.4110, 0.4343]**	**0.4226**	**[0.4141, 0.4310]**
2008:1	51288	0.4025	[0.3911, 0.4139]	0.4025	[0.3967, 0.4083]
2008:2	51296	0.4177	[0.4079, 0.4276]	0.4177	[0.4112, 0.4241]
2008:3	51292	0.4229	[0.4115, 0.4343]	0.4229	[0.4153, 0.4304]
2008:4	**51300**	**0.4088**	**[0.3923, 0.4252]**	**0.4086**	**[0.3962, 0.4210]**
2009:1	51285	0.3785	[0.3645, 0.3926]	0.3785	[0.3714, 0.3856]
2009:2	45094	0.3821	[0.3711, 0.3931]	0.3822	[0.3762, 0.3881]
2009:3	51300	0.3887	[0.3824, 0.3950]	0.3887	[0.3829, 0.3945]
2009:4	**51309**	**0.3840**	**[0.3774, 0.3906]**	**0.3840**	**[0.3780, 0.3900]**

Table 5. The official Gini coefficient for income inequality among Russian households

Year	1992	1995	2000	2004	2005	2006	2007	2008	2009
Gini coefficient	0.289	0.387	0.395	0.409	0.409	0.416	0.423	0.422	0.422

Source: Rosstat, the Federal State Statistics Service of Russian Federation.

Figure 1. Gini coefficients for income distribution among Russian households and its parts on the left and the right from the mode

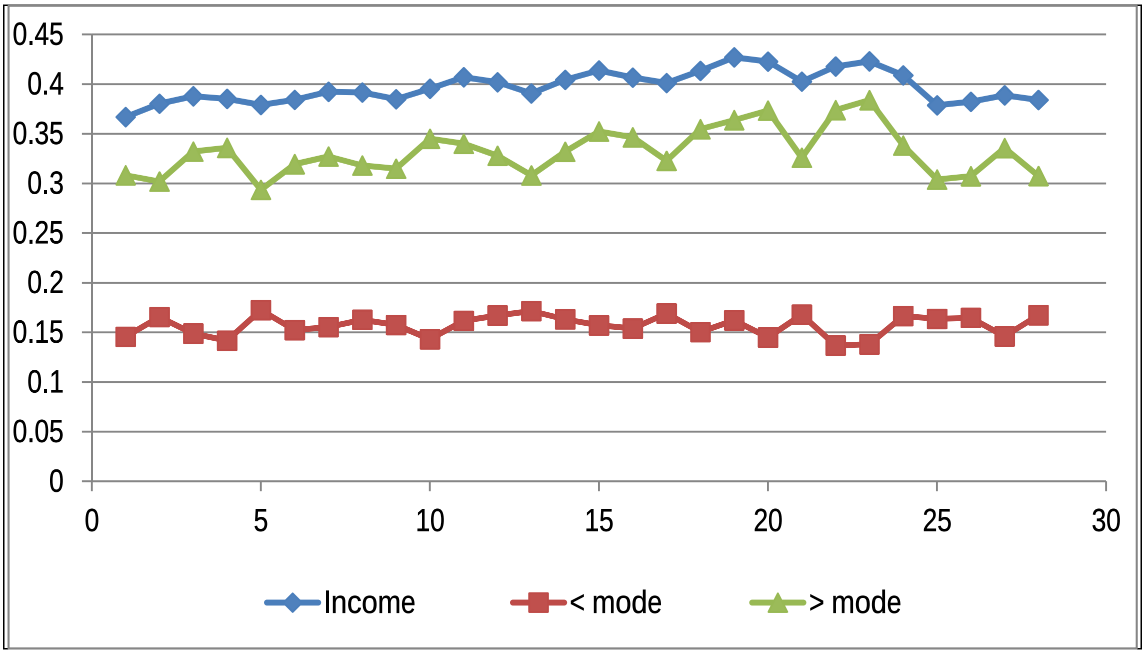


Figure 2. Maximum likelihood estimates of the parameters (k_1, k_2) of the double Pareto model and the parameters (k_{tr1}, k_{tr2}) of the truncated double Pareto model for income distribution among Russian households

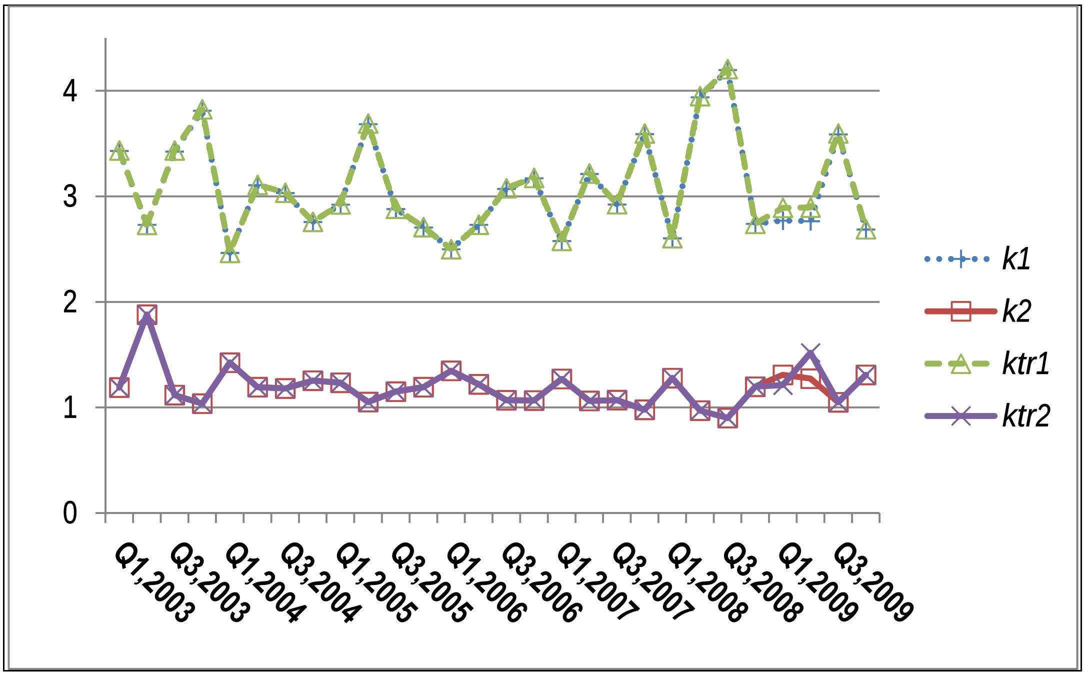


Figure 3. Average $\bar{G} = \frac{\sum_{i=1}^{q} \bar{G}_i}{q}$ of the group empirical Gini coefficients and the robust *t*-statistic based 95% confidence intervals (q=4) for the Gini coefficients for income distribution among Russian households, 4th quarters of 2003-2009

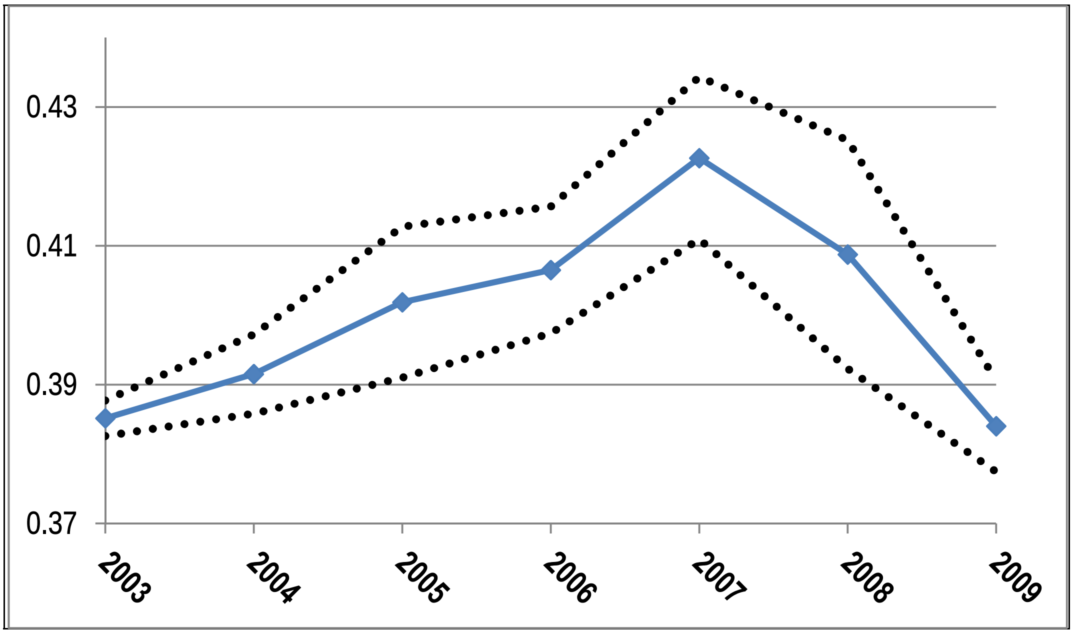
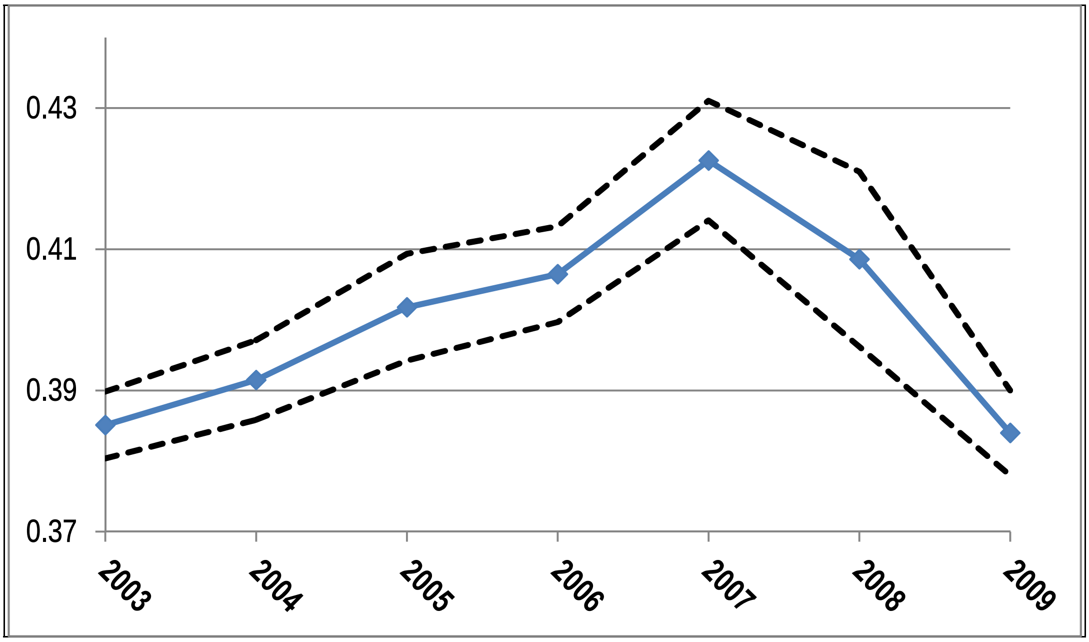


Figure 4. Average $\bar{G} = \frac{\sum_{i=1}^{q} \bar{G}_i}{q}$ of the group empirical Gini coefficients and the robust *t*-statistic based 95% confidence intervals (q=8) for the Gini coefficients for income distribution among Russia households, 4th quarters of 2003-2009



B. Inequality measures

As usual, throughout the appendices, given a r.v. (e.g., income level or its logarithm) I, we denote by $\mu_I = E[I]$ and $\sigma_I^2 = E(I-\mu_I)^2$ its mean and variance, respectively. In addition, as usual, for a sample $I_1,\dots,I_q$, $\bar{I} = q^{-1}\sum_{j=1}^{q} I_j$ will denote the sample mean of the observations I_j*'s* and $s_I^2 = (q-1)^{-1}\sum_{j=1}^{q} E(I_j-\bar{I})^2$ will denote their sample variance.

In this appendix, we review the definitions of the risk, inequality, poverty and concentration measures considered in the paper. The detailed discussions of the properties of these and other measures are available, for instance, in Section 13.F in Marshall et al., 2011, Cowell & Flachaire, 2007, Davidson & Flachaire, 2007, and references therein.

Variance: The variance σ_I^2 and the sample variance s_I^2 are standard examples of population and empirical measures of dispersion or spread of I around its mean μ_I.

Coefficient of variation: The population coefficient of variation is the normalized standard deviation defined by $CV_I = \sigma_I/\mu_I$. The commonly used natural estimator of the coefficient CV_I is given by the empirical coefficient of variation $\widehat{CV}_I = {}^{s_I}/_{\bar{I}_q}$.

Variance and coefficient of variation of logarithms and logarithmic variance: The variance and the coefficient of variation of logarithms have the form $\sigma_Y^2 = E(Y-EY)^2, CV_Y = {}^{\sigma_Y}/_{\mu_Y}$, where $Y = \log(I)$. The commonly used estimators of these measures are provided by the sample variance s_Y^2 and the empirical coefficient of variation $\widehat{CV}_Y = {}^{s_Y}/_{\bar{Y}_q}$ for the logarithms $Y_j = log(I_j)$ of observations I_j. A related measure is the logarithmic variance defined by

$$\mathcal{L}_{LV} = E\left[log\left(\frac{I}{\mu_I}\right)\right]^2 = E(Y^2) - 2Ylog(\mu_I)+[log(\mu_I)]^2.$$

Generalized entropy (GE) measures, $\alpha \neq 0,1$:

$$\mathcal{L}_E^{\alpha} = \frac{1}{\alpha(\alpha-1)}\left(\frac{EI^{\alpha}}{\mu_I} - 1\right). \tag{B1}$$

Mean logarithmic deviation (MLD) is a limiting case of GE measures as $\alpha \to 0$:

$$MLD = \mathcal{L}_E^0 = \log(\mu_I) - \mu_I. \tag{B2}$$

Theil measure is a limiting case of GE indices as $\alpha \to 1$:

$$\mathcal{L}_E^1 = \frac{E[I\log(I)]}{\mu_I} - \log(\mu_I). \tag{B3}$$

Gini coefficient: The Gini coefficient has the form $Gini_I = E[|I - I'|]$, where I' is an independent copy of I. The empirical Gini coefficient is calculated using the formulas

$$\widehat{Gini}_I = \frac{1}{2q^2\bar{I}}\sum_{i=1}^{q}\sum_{j=1}^{q}|I_i - I_j| = 1 - \frac{1}{2q^2\bar{I}}\sum_{i=1}^{q}\sum_{j=1}^{q}\min(I_i, I_j) = 1 + \frac{1}{q} - \frac{1}{2q^2\bar{I}}\sum_{j=1}^{q} iI_{[i]},$$

where $I_{[1]} \geq \ldots \geq I_{[q]}$ are the components of $(I_1, \ldots, I_q)$ arranged in non-increasing order.

C. *t*-statistic based correlation and heterogeneity robust inference

C.1. Description of the approach

Suppose we want to do inference on a scalar parameter β of an econometric model in a large data set with n observations. For a wide range of models and estimators $\hat{\beta}$ it is known that $\sqrt{n}(\hat{\beta} - \beta) \rightarrow_d N(0, \sigma^2)$ as $n \rightarrow \infty$, where "$\rightarrow_d$" denotes convergence in distribution. Suppose further that the observations exhibit correlations of largely unknown form. If such correlations are pervasive and pronounced enough, then it will be very challenging to consistently estimate σ^2, and inference procedures for β that ignore the sampling variability of a candidate consistent estimator $\hat{\sigma}^2$ will have poor finite sample properties.

Ibragimov & Müller (2010) propose the following general approach to robust inference about the parameter β under heterogeneity and correlation of a largely unknown form. Consider a partition the original data set into $q \geq 2$ groups, with n_j observations in group j, and $\sum_{j=1}^{q} n_j = n$. Denote by $\hat{\beta}_j$ the estimator of β using observations in group j only. Suppose the groups are chosen such that $\sqrt{n}(\hat{\beta}_j - \beta) \rightarrow_d N(0, \sigma_j^2)$ for all j, and, crucially, such that $\sqrt{n}(\hat{\beta}_i - \beta)$ and $\sqrt{n}(\hat{\beta}_j - \beta)$ are asymptotically independent for $i \neq j$: this amounts to the convergence in distribution

$$\sqrt{n}(\hat{\beta}_1 - \beta, \ldots, \hat{\beta}_q - \beta)' \rightarrow_d N(0, \mathrm{diag}(\sigma_1^2, \ldots, \sigma_q^2)), \qquad \max_{1 \leq j \leq q} \sigma_j^2 > 0, \tag{C1}$$

and $\{\sigma_j^2\}_{j=1}^{q}$ are, of course, unknown. The asymptotic Gaussianity of $\sqrt{n}(\hat{\beta}_j - \beta)$, $j = 1, \ldots, q$, typically follows from the same reasoning as the asymptotic Gaussianity of the full sample estimator $\hat{\beta}$. The argument for asymptotic independence of $\hat{\beta}_i$ and $\hat{\beta}_j$ for $i \neq j$, on the other hand, depends on the choice of groups and the details of the application (see Section 4 in Ibragimov & Müller 2010 for the discussion of such arguments for a number of important econometric models, including time series, panel, clustered and spatially correlated settings, and Section 3 in the paper for the case of inference on income inequality measures).

As discussed in Ibragimov & Müller (2010), one can perform an asymptotically valid test of level α, $\alpha \leq 0.05$ of H_0: $\beta = \beta_0$ against H_1: $\beta \neq \beta_0$ by rejecting H_0 when $|t_\beta|$ exceeds the $(1-\alpha/2)$ percentile of the Student-*t* distribution with $q-1$ degrees of freedom, where t_β is the usual *t*-statistic

$$t_\beta = \sqrt{q}\,\frac{\bar{\hat{\beta}}-\beta_0}{s_{\hat{\beta}}} \tag{C2}$$

with $\bar{\hat{\beta}} = q^{-1}\sum_{j=1}^{q}\hat{\beta}_j$, the sample mean of the group estimators $\hat{\beta}_j$, j = 1,..., q, and $s_{\hat{\beta}}^2 = (q-1)^{-1}\sum_{j=1}^{q}(\hat{\beta}_j - \bar{\hat{\beta}})^2$, the sample variance of $\hat{\beta}_j$, j = 1,..., q.

In other words, the usual *t*-tests can be used in the presence of asymptotic heteroskedasticity in group estimators as long the level of the tests is not greater than the typically used 5% threshold. As discussed in Ibragimov & Müller (2010), the *t*-statistic approach provides a number of important advantages over the existing methods. In particular, it can be employed when data are potentially heterogeneous and correlated in a largely unknown way. In addition, the approach is simple to implement and does not need new tables of critical values. The assumptions of asymptotic normality for group estimators in the approach are explicit and easy to interpret, in contrast to conditions that imply validity of alternative procedures. Furthermore, as shown in Ibragimov & Müller (2010), the *t*-statistic based approach to robust inference efficiently exploits the information contained in these regularity assumptions, both in the small sample settings (uniformly most powerful scale invariant test against a benchmark alternative with equal variances) and also in the asymptotic frameworks. It is important to emphasize that the asymptotic efficiency results for *t*-statistic based robust inference further imply that it is not possible to use data dependent methods to determine the optimal number of groups q to be used in the approach when the only assumption imposed on the data generating process is that of asymptotic normality for the group estimators $\hat{\beta}_j$. The numerical results presented in Ibragimov & Müller (2010) demonstrate that, for many dependence and heterogeneity settings considered in the literature and typically encountered in practice for time series, panel, clustered and spatially correlated data, the choice q = 8 or q = 16 leads to robust tests with attractive finite sample performance.

One should also note that, as discussed in Ibragimov & Müller (2010), the *t*-statistic approach described provides a formal justification for the widespread Fama-MacBeth method for inference in panel regressions with heteroskedasticity (see Fama & MacBeth 1973). In the approach, one estimates the regression separately for each year, and then tests hypotheses about the coefficient of interest using the *t*-statistic of the resulting yearly coefficient estimates. The Fama-MacBeth approach is a special case of the *t*-statistic based approach to inference, with observations of the same year collected in a group.

In addition, the same approach remains valid under deviations from normality as in the case of heavy-tailed models, as long as the estimators $\hat{\beta}_j$, j = 1,..., q, are asymptotically independent and converge (at an arbitrary rate) to scale mixtures of normals. Namely, the approach is asymptotically valid if

$$\left\{m_n(\hat{\beta}_j - \beta)\right\}_{j=1}^{q} \to_d \left\{Z_j V_j\right\}_{j=1}^{q} \tag{C3}$$

for some real sequence m_n, where $Z_j \sim i.i.d.\ N(0,1)$, the random vector $\left\{V_j\right\}_{j=1}^{q}$ is independent of the vector $\left\{Z_j\right\}_{j=1}^{q}$ and $\max_j |V_j| > 0$ almost surely. The class of limiting scale mixtures of normals in (C3) is a rather large class of distributions: it includes, for instance, the Student-*t* distributions with arbitrary degrees of freedom (including the Cauchy distribution), the double exponential distribution, the logistic distribution and all symmetric stable distributions that typically arise as distributional limits of estimators in econometric models under heavy-tailedness with infinite variances.

The robust approach to asymptotic inference proposed in Ibragimov & Müller (2010) relies on the following powerful result on small sample properties of the *t*-statistic in heterogeneous normal observations due to Bakirov & Székely (2005) (see also the independent proof of the result in the working paper version of Ibragimov & Müller, 2010).

Let X_j, $j=1,\dots,q$, with $q \geq 2$, be independent Gaussian random variables with common mean $E[X_j]=\mu$ and variances $V[X_j]=\sigma_j^2$. Consider the usual *t*-statistic for the hypothesis test H_0: $\mu = 0$ against the alternative $H_a : \mu \neq 0$:

$$t = \sqrt{q}\,\frac{\bar{X}}{s_X}. \tag{C4}$$

If the variances σ_j^2 are the same: $\sigma_j^2 = \sigma^2$ for all j, by definition, the critical value cv of $|t|$ is given by the appropriate percentile of the distribution of a Student-*t* distributed random variable T_{q-1} with $q-1$ degrees of freedom.

The case of equal variances is extremal for the *t*-statistic in (1) in the following sense (see Bakirov & Székely, 2005, and Theorem 1 in Ibragimov & Müller, 2010). Let $cv_q(\alpha)$ be the critical value of the usual two-sided *t*-test of H_0 against H_a of level $\alpha \leq 0.05$: $P(|T_{q-1}| > cv_q(\alpha))=\alpha$. Then, for all $q \geq 2$,

$$\sup_{\{\sigma_1^2,\dots,\sigma_q^2\}} P(|t|>cv_q(\alpha)|H_0) = P(|t| > cv_q(\alpha)|H_0, \sigma_1^2 = \dots = \sigma_q^2) = P(|T_{q-1}| > cv_q(\alpha)). \tag{C5}$$

The conservativeness result in (C5) does not hold for 10% level with $\alpha = 0.1$.

The conservativeness properties of *t*-statistic given by (C5) imply analogous results for the tail probabilities of self-normalized sums

$$S_q = \sum_{j=1}^{q} X_j \Big/ \left(\sum_{j=1}^{q} X_j^2\right)^{1/2} \tag{C6}$$

and their squares using the equality (see, for instance, Edelman, 1990)

$$P(|t| > y) = P\left(s_q^2 > \frac{qy^2}{y^2+q-1}\right) \qquad \text{(C7)}$$

for all $y > 0$.

C.2. Finite sample properties and comparisons with alternative inference procedures

This section provides numerical results on finite sample performance of the t-statistic based approach to robust inference on commonly used inequality measures in comparison with the alternative procedures. In particular, we focus on the comparison of the error in rejection probabilities (ERP) of the t-statistic based tests on inequality indices with those of the standard asymptotic and bootstrap tests for inequality and non-standard bootstrap inference procedures, including the m out of n bootstrap (also known as the *moon* bootstrap) and a semiparametric bootstrap (see Cowell & Flachaire, 2007 and Davidson & Flachaire, 2007). We also provide analogous comparisons of the finite sample properties of methods discussed in the paper with those of asymptotic tests based on semiparametric estimation of the income distribution (semiparametric inequality measures) discussed in Cowell & Flachaire (2007). As indicated in Cowell & Flachaire (2007), outliers and heavy-tailedness in income distribution have dramatic effects on performance of empirical inequality measures, even when the standard bootstrap procedures are employed. According to the results presented in Cowell & Flachaire (2007), semiparametric inference approaches, such as asymptotic tests based on semiparametric inequality measures and semiparametric bootstrap, can greatly improve the performance of many commonly used empirical inequality indices.

As in the case of non-parametric and semiparametric asymptotic and bootstrap procedures considered in Cowell & Flachaire (2007), the data used in the analysis of the ERP reported in this section are simulated using the Singh-Maddala, Pareto and log-normal cdf's that are widely used in modeling observed income distributions (see the discussion and references in Cowell & Flachaire, 2007, and Davidson & Flachaire, 2007).

R.v.'s I (e.g., income or wealth) with Singh-Maddala distribution satisfy $P(I > y) = \frac{1}{(1+ay^b)^c}$, $y>0$ with parameters $a, b, c > 0$ and thus follow power law (1) with the tail index $\zeta = bc$. The true values of the GE measures for Singh-Maddala income distributions are obtained using definition (B1) and the following formulas for moments of r.v.'s I with such distributions (see Section 2.1 in Cowell & Flachaire 2007): $E[I^\alpha] = a^{-\alpha/b}\frac{\Gamma(1+\alpha b^{-1})\Gamma(c-\alpha b^{-1})}{\Gamma(c)}$, where $\Gamma(y)=\int_0^\infty t^{y-1}\exp(-t)\,dt,\ \ y \in \mathbf{R}$, is the Gamma function (in particular, $E(I) = a^{-1/b}\frac{\Gamma(1+b^{-1})\Gamma(c-b^{-1})}{\Gamma(c)}$). The true values of the mean logarithmic deviation and the Theil measure for Singh-Maddala distribution are found from (B2) and

(B3), together with the following formulas:

$$E[I\log(I)] = E[I]\, b^{-1}[\gamma(b^{-1}+1) - \gamma(c-b^{-1}) - \log(a)],$$

$$E[\log(I)] = b^{-1}[\gamma(1) - \gamma(c) - \log(a)],$$

where $\gamma(y) = \frac{1}{\Gamma(y)}\frac{d\log(\Gamma(y))}{dy}$ is the digamma function.

R.v.'s I with Pareto distributions satisfy (1) with the exact equality for $y \geq y_0$, where $y_0 = C^{1/\zeta}$:

$$P(I>y) = \frac{C}{y^{\zeta}},\ y \geq y_0. \tag{C8}$$

The Theil index $\mathcal{L}^1$ and the mean logarithmic deviation $\mathcal{L}^0$ for Pareto income distribution are given by $\mathcal{L}^1 = \frac{1}{\zeta-1} + \log\left(\frac{\zeta-1}{\zeta}\right)$ and $\mathcal{L}^0 = -\frac{1}{\zeta} - \log\left(\frac{\zeta-1}{\zeta}\right)$ (see Section 4 in Cowell & Flachaire 2007).

The density of the log-normal distribution is given by $\frac{1}{\sqrt{2\pi}\sigma y}\exp\left[-\frac{1}{2\sigma^2}(\log(y)-\mu)^2\right]$. The tails of log-normal distributions are thinner than those of power laws (1): in particular, all power moments of r.v.'s I with log-normal distributions are finite: $E[|I|^p] < \infty$ for all $p > 0$. However, similar to power laws, the moment generating function of I is infinite in any neighborhood of zero: $E[\exp(cI)] = \infty$ for all $c > 0$. In part because of this reason, log-normal distributions are difficult to distinguish from power laws in empirical applications (see the discussion in Perline, 2005). The Theil index and the mean logarithmic deviation for log-normal income distribution are both equal to $\mathcal{L}^1 = \mathcal{L}^0 = \sigma^2/2$.

In simulations presented below, we use the same parameters for Singh-Maddala, Pareto and log-normal distributions as in Section 4 of Cowell & Flachaire (2007). The parameters for the Singh-Maddala distributions are $a = 100$, $b = 2.8$, and $c = 0.7, 1.2, 1.7$.[8] The corresponding tail indices $\zeta = bc$ in asymptotic relation (1) for these distributions equal to, respectively, $\zeta = 1.96$ (implying finite first moments and infinite variances), $\zeta = 3.36$ (finite variances and infinite fourth moments) and $\zeta = 4.76$ (finite fourth moments but infinite moments of order greater than ζ). For the above choice of the parameters $a = 100$, $b = 2.8$ and $c = 1.7$ as in Section 3 in Cowell & Flachaire (2007) and Table C.1 in this section, the true values of the inequality measures are given by (see Cowell & Flachaire 2007) $\mathcal{L}_E^2 = 0.1620$, $\mathcal{L}_E^1 = 0.1401$, $\mathcal{L}_E^{0.5} = 0.1397$, $\mathcal{L}_E^0 = 0.1460$, $\mathcal{L}_E^{-1} = 0.1898$, $\mathcal{L}_E^{-2} = 0.3866$, $\mathcal{L}_{LV} = 0.3321$ and $\mathcal{L}_{Gini} = 0.2887$.

The simulations for Pareto distributions (C8) use the threshold value $y_0 = 0.1$ and the tail index parameters ζ equal to $\zeta = 1.5$ and 2 (finite means and infinite variances) and $\zeta = 2.5$ (finite

[8] As indicated in Cowell & Flachaire (2007), the choice of the parameter values $a = 100$, $b = 2.8$ and $c = 1.7$ is motivated by the fact that the Singh-Maddala distribution with these parameters closely approximates the net income distribution of German households, up to a scale factor.

variances and infinite fourth moments).[9]

The simulations for log-normal distributions use $\mu = -2$ and $\sigma = 1,\ 0.7,\ 0.5$.

As in Cowell & Flachaire (2007), we first focus on the analysis of performance of the *t*-statistic based tests on inequality measures under Singh-Maddala income distributions with parameters a = 100, b = 2.8 and c = 1.7. Table C.1 presents the ERP of the *t*-statistic based tests with q = 2,4 at nominal level 0.05, that is, the difference between the actual and nominal probabilities of rejection, for different GE measures, the Gini index and the logarithmic variance.

The comparison of the results on the of ERP of the *t*-statistic based tests on inequality measures reported in Table C.1 with those in Figure 7 in Cowell & Flachaire (2007) indicates that the size properties of the *t*-statistic based robust tests with q = 2 and q = 4 in small size are uniformly better than those of the asymptotic tests.

Table C1. ERP of the *t*-statistic based robust tests on inequality measures:

Singh-Maddala income distribution with the parameters a = 100, b = 2.8 and c = 1.7

N	**Generalized entropy (GE) $\mathcal{L}_E^\alpha$**					**Gini**	**Logarithmic variance**
	***α*= –1**	***α*=0**	***α*=0.5**	***α*=1**	***α*=2**		
***q*=2**							
500	0.0073	0.0029	0.0028	0.0026	0.0156	0.0004	–0.0035
1000	0.0008	0.0019	–0.0012	–0.0008	0.0043	–0.0011	–0.0028
2000	0.0024	0.0011	–0.0025	–0.0009	0.0051	–0.0061	–0.0005
3000	0.0016	0.0023	–0.0017	–0.0025	0.0022	–0.0018	–0.0005
4000	0.0004	–0.0005	–0.0001	0.0017	0.0088	0.0020	0.0018
5000	–0.0006	0.0001	–0.0024	–0.0006	0.0009	0.0008	–0.0004
6000	–0.0008	–0.0022	0.0034	0.0002	0.0053	–0.0034	0.0017
7000	–0.0028	0.0014	0.0000	0.0014	0.0010	–0.0028	0.0011
8000	0.0004	–0.0033	0.0006	–0.0006	0.0047	–0.0005	0.0002
9000	0.0041	0.0052	0.0037	–0.0031	0.0074	–0.0004	–0.0025
10000	–0.0001	0.0013	–0.0025	–0.0013	0.0048	–0.0012	–0.0036
***q*=4**							
500	0.0154	0.0093	0.0076	0.0182	0.0580	0.0080	0.0080
1000	0.0098	0.0050	0.0076	0.0048	0.0416	0.0032	0.0018
2000	0.0045	0.0024	0.0037	0.0093	0.0274	0.0033	0.0004
3000	0.0035	0.0017	0.0032	0.0060	0.0213	0.0060	0.0014
4000	0.0096	–0.0041	0.0043	–0.0009	0.0258	0.0003	0.0000
5000	0.0015	0.0016	0.0040	0.0011	0.0210	0.0010	0.0040
6000	–0.0010	0.0060	–0.0008	0.0046	0.0178	0.0011	0.0015
7000	0.0076	–0.0002	0.0036	0.0008	0.0187	0.0015	–0.0033
8000	0.0012	0.0005	0.0002	0.0024	0.0154	0.0016	0.0030
9000	0.0050	–0.0011	0.0001	0.0048	0.0191	0.0021	0.0023
10000	–0.0007	0.0003	–0.0038	0.0036	0.0094	0.0018	–0.0021

[9] The corresponding values of the constant $C = y_0^\zeta$ in (C8) equal to, respectively, C = 0.0316, 0.01 and C = 0.0032.

In addition, the finite sample properties of the t-statistic based tests on inequality measures, especially that with $q = 2$, are at least comparable to and in many cases dominate the size properties of the computationally expensive alternatives, including the standard and non-standard bootstrap methods, as well as those of the asymptotic tests based on semiparametric inequality measures and semiparametric boostrap (see Figures 8-11 in Cowell & Flachaire 2007 and the discussion in Section 2 therein).

Similar to the analysis of the alternative inference procedures in Cowell & Flachaire (2007), the following Tables C2 and C3 provide the results on finite sample performance of the t-statistic based robust tests on inequality for different parameters in the Singh-Maddala distributions for incomes as well as Pareto and log-normal distributions. As in Cowell & Flachaire (2007), the results are provided for the ERP of the t-statistic based tests on the Theil and mean logarithmic deviation (MLD) measures (that is, the generalized entropy measures with $\alpha = 1$ and $\alpha = 0$, respectively).

Comparison of the ERP of the t-statistic based tests on the Theil measure and the mean logarithmic deviation in Tables C2 and C3 with the corresponding results in Tables 5 and 6 in Cowell & Flachaire (2007) for the alternative procedures leads to the following conclusions. In essentially all choices of the sample sizes and the parameter values for the distributions considered, the finite sample properties of the t-statistic based tests with $q = 2$ and $q = 4$ on the Theil index and the mean logarithmic deviation are much better than those of the alternative procedures, including the asymptotic inference methods (where the better finite sample performance of the t-statistic based robust tests is especially pronounced), standard, *moon* and semiparametric bootstrap tests as well as the asymptotic tests with semiparametric inequality measures. In addition, according to the results in Tables C2 and C3, the choice of the smaller number of blocks $q = 2$ is to be preferred, in terms of the finite sample size performance of the t-statistic based tests, to $q = 4$. According to the unreported simulation results, the choice of $q = 2$ leads to better performance of the t-statistic based tests comparing to the number of blocks greater than 4 in the samples considered. The above conclusions on the number of blocks complement those in Ibragimov & Müller (2010) where, as discussed in Appendix C.1, the numerical results indicate the best finite sample performance for the number of groups $q = 8$ or $q = 16$ for many dependence and heterogeneity settings considered in the literature and typically encountered in applications for time series, panel, clustered and spatially correlated data.

Table C2. ERP of the *t*-statistic based tests on the MLD measure (GE measure with α = 0): Singh-Maddala income distributions with the parameters a = 100, b = 2.8 and c; Pareto income distributions with the parameters y_0 =1 and ζ ; and log-normal income distributions with the parameters μ=–2 and σ

N	**Singh-Maddala**			**Pareto**			**Log-normal**		
	***c*=0.7**	***c*=1.2**	***c*=1.7**	**ζ=1.5**	**ζ=2**	**ζ=2.5**	**σ=1**	**σ=0.7**	**σ=0.5**
***q*=2**									
500	0.0144	0.0018	0.0025	0.0416	0.0145	0.0124	0.0037	0.0027	0.0040
1000	0.0141	0.0041	0.0031	0.0387	0.0099	0.0091	0.0004	–0.0045	0.0044
2000	0.0114	0.0023	0.0003	0.0283	0.0080	0.0067	–0.0004	–0.0006	0.0026
3000	0.0066	0.0004	0.0002	0.0293	0.0075	0.0011	–0.0022	0.0031	0.0014
4000	0.0081	0.0004	0.0022	0.0310	0.0055	–0.0018	–0.0007	–0.0027	–0.0034
5000	0.0012	–0.0004	–0.0006	0.0266	0.0035	0.0035	–0.0004	0.0016	–0.0006
***q*=4**									
500	0.0684	0.0125	0.0131	0.1754	0.0835	0.0516	0.0114	0.0084	0.0084
1000	0.0519	0.0091	0.0053	0.1553	0.0637	0.0301	0.0050	0.0015	0.0058
2000	0.0369	0.0066	0.0018	0.1282	0.0448	0.0232	0.0037	0.0037	0.0004
3000	0.0344	0.0030	–0.0004	0.1201	0.0387	0.0181	0.0014	0.0017	–0.0019
4000	0.0312	0.0030	0.0029	0.1093	0.0361	0.0177	0.0045	0.0010	–0.0008
5000	0.0316	0.0026	–0.0007	0.1115	0.0321	0.0182	0.0012	–0.0008	–0.0017

Table C3. ERP of the *t*-statistic based test on the Theil measure (GE measure with α = 1): Singh-Maddala income distributions with the parameters a = 100, b = 2.8 and c; Pareto income distributions with the parameters y_0 =1 and ζ; and log-normal income distribution with the parameters μ=–2 and σ

N	**Singh-Maddala**			**Pareto**			**Lognormal**		
	***c*=0.7**	***c*=1.2**	***c*=1.7**	**ζ=1.5**	**ζ=2**	**ζ=2.5**	**σ=1**	**σ=0.7**	**σ=0.5**
***q*=2**									
500	0.0502	0.0138	0.0037	0.1211	0.0475	0.0221	0.0033	0.0033	0.0020
1000	0.0387	0.0069	−0.0021	0.1033	0.0362	0.0191	0.0018	0.0055	0.0013
2000	0.0301	0.0053	0.0016	0.0960	0.0298	0.0114	0.0038	0.0010	0.0009
3000	0.0322	0.0060	0.0000	0.0858	0.0293	0.0136	0.0008	0.0022	0.0000
4000	0.0264	0.0041	0.0006	0.0898	0.0231	0.0110	0.0028	−0.0020	0.0001
5000	0.0244	0.0040	−0.0026	0.0827	0.0183	0.0065	0.0016	0.0018	−0.0010
***q*=4**									
500	0.2103	0.0480	0.0186	0.4415	0.1948	0.1160	0.0428	0.0178	0.0119
1000	0.1647	0.0348	0.0138	0.3841	0.1605	0.0945	0.0227	0.0111	0.0043
2000	0.1437	0.0228	0.0030	0.3532	0.1368	0.0669	0.0153	0.0055	−0.0004
3000	0.1288	0.0187	0.0052	0.3325	0.1260	0.0601	0.0044	0.0016	0.0001
4000	0.1201	0.0168	0.0067	0.3149	0.1186	0.0555	0.0077	0.0051	−0.0040
5000	0.1080	0.0127	0.0013	0.3187	0.1055	0.0414	0.0038	0.0035	−0.0003

C.3. Small sample properties of inequality indices based on self-normalized sums

This section presents some theoretical results that indicate the connection of the small sample conservativeness properties of t-statistics discussed in Appendix C.1 to income inequality measures and other economic indicators such as the coefficient of variation.

Consider a sample of observations (e.g., income levels) $X_1, \ldots, X_q, q \geq 2$. Let, as in Section 2 and Appendix C.1, $cv_q(\alpha)$ denote the $(1-\alpha/2)$-quantile of Student-t distribution with $(q-1)$ degrees of freedom: $P(|T_{q-1}| > cv_q(\alpha)) = \alpha$.

Representations similar to t-statistic $t = \sqrt{q}\bar{X}/s_X$, and self-normalized sums $S_q = \frac{\sum_{j=1}^{q} X_j}{\sqrt{\sum_{j=1}^{q} X_j^2}}$ in (C4) and (C6) hold for a number of variables of interest in economics and finance, including, for instance, one of the widely used inequality measures, the empirical coefficient of variation $\widehat{CV} = \frac{s_X}{\bar{X}} = \sqrt{q}/t$. [10] These representations, together with the conservativeness results for t-statistics and self-normalized sums given by (C5) and (C7) imply similar results for the tail probabilities of the empirical coefficient of variation $\widehat{CV}$, and a number of other important indicators in economics and finance. The conservativeness results for the empirical inequality measures such as $\widehat{CV}$ and their analogues for transformations (such as logarithms) of the observations X_j provide comparisons between the tail probabilities and the cdf's of these measures under heterogeneity and heavy-tailedness and those in the standard homogeneous Gaussian case.

Below, $Y_j = \log(I_j)$, $j=1, \ldots, q$, denote the logarithms of observations on income levels $I_1, \ldots, I_q > 0$. In addition, $\tilde{Y}_1, \ldots, \tilde{Y}_q$ denote the i.i.d. standard normal r.v.'s: $\tilde{Y}_j \sim N(0,1)$.

Proposition 1. *If $Y_1, \ldots, Y_q$ are independent heterogenous normal r.v.'s $Y_j \sim N(0, \sigma_j^2)$ (so that the income levels I_j are log-normal with $\mu=0$ and heterogeneous parameters σ_j) or are scale mixtures of normals (for instance, independent not necessarily identically distributed stable r.v.'s), then*

$$P(0 < \widehat{CV}_Y < y) \leq P(0 < \widehat{CV}_{\tilde{Y}} < y) \tag{C9}$$

$$P(|\widehat{CV}_Y| < y) \leq P(|\widehat{CV}_{\tilde{Y}}| < y) \tag{C10}$$

for all $y < 1/(cv_{q-1}(0.05)\sqrt{q})$. *In general, inequalities* (C8), (C9) *do not hold for* $y < 1/(cv_{q-1}(0.1)\sqrt{q})$.

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[10] Similar representations also hold for the estimators of Sharpe ratio SR for excess returns X_j, $j = 1,\ldots,q$. In addition, this is the case for the Herfindahl-Hirschman Index of market concentration that has the form $HHI = \sum_{j=1}^{q} X_j^2 / \left(\sum_{j=1}^{q} X_j\right)^2$ and is, thus, the inverse of the square of the self-normalized ratio in (C6) for firm sizes X_j, $j = 1,\ldots, q$. The representations also hold, for instance, for commonly used sample split prediction test statistics employed in testing for time series stationarity (see Loretan & Phillips 1994, and references therein).

Inequalities (C9)-(C10) imply that homogeneity and thin-tailedness (such as normality) are likely to reduce the inequality and disparity, as measured by the coefficient of variation, in the region of their small values. However, in general, this does not hold, that may be viewed as an indicator that the coefficient of variation is a poor measure of inequality for some parts of the income or wealth distribution, including the middle and high income and wealth ranges.

D. Double Pareto distribution and truncated double Pareto distribution

The pdf of double Pareto distribution (see Toda, 2012) has the form

$$f(y) = \begin{cases} \frac{k_1 k_2}{k_1+k_2}\frac{1}{M}\left(\frac{y}{M}\right)^{k_1-1}, & (0 \le y < M), \\ \frac{k_1 k_2}{k_1+k_2}\frac{1}{M}\left(\frac{y}{M}\right)^{-k_2-1}, & (y \ge M), \end{cases} \tag{D1}$$

where $M > 0$ is the location parameter (the mode) and $k_1,\ k_2 > 0$ are the shape parameters.

The cdf of a r.v. Y with double Pareto density (D1) is

$$F(y) = \mathrm{P}(Y \le y) = \begin{cases} \frac{k_2}{k_1+k_2}\left(\frac{y}{M}\right)^{k_1}, & (0 \le y < M), \\ 1 - \frac{k_1}{k_1+k_2}\left(\frac{y}{M}\right)^{-k_2}, & (y \ge M). \end{cases} \tag{D2}$$

Thus, the tail probabilities of the double Pareto distribution are given by, for $y \ge M$,

$$P(Y > y) = \frac{k_1}{k_1+k_2}\left(\frac{y}{M}\right)^{-k_2}, \tag{D3}$$

and, consequently, the double Pareto distribution follows power law (1) with the tail index $\zeta = k_2$. Furthermore, as is easy to see, the conditional tail probability $P(Y>y\,|\,Y>M)$ is Pareto (C8) with $\zeta = k_2$, $y_0=M$ and the equality in (1) for $y \ge M$: $P(Y > y\,|\,Y > M) = \left(\frac{y}{M}\right)^{-k_2}$.

The expectation of a r.v. Y with double Pareto distribution (D1)-(D2) is finite for $k_2>1$ (see the discussion of general power laws in Section 1.2) and, in that case, is given by the formula

$$\mu = E[Y] = M\frac{k_1 k_2}{k_1+k_2}\left[\int_0^1 y^{k_1}\,dy + \int_1^\infty y^{-k_2}\,dy\right] = \frac{k_1 k_2}{(k_1+1)(k_2-1)}M.$$

The median of Y equals to $me = M\left(\sqrt[k_2]{\frac{2k_1}{k_1+k_2}}\right)$.

The Gini coefficient of the double Pareto distribution is

$$G = \frac{1}{\mu}\int_0^\infty F(y)\big(1-F(y)\big)dy = \frac{2k_1^2+2k_1k_2+2k_2^2+k_1-k_2}{(k_1+k_2)(2k_1+1)(2k_2-1)}.$$

Let us now introduce a truncated analogue of the double Pareto distribution concentrated on the interval $[0, b]$, where $b \ge M$ is a truncation bound. The cdf of the truncated double Pareto distribution is given by $F_b(y)=\frac{F(y)}{F(b)}=\frac{F(y)}{1-\frac{k_1}{k_1+k_2}\left(\frac{b}{M}\right)^{-k_2}}=\frac{k_1+k_2}{k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}}F(y),\ y\in[0,b]$, or

$$F_b(y)=\begin{cases}\dfrac{k_2}{k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}}\left(\dfrac{y}{M}\right)^{k_1}, & (0\le y<M)\\ \dfrac{k_1+k_2-k_1\left(\frac{y}{M}\right)^{-k_2}}{k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}}, & (M\le y\le b)\end{cases}.$$

The corresponding pdf is

$$f_b(y)=\begin{cases}\dfrac{k_1k_2}{M\left(k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}\right)}\left(\dfrac{y}{M}\right)^{k_1-1}, & (0\le y<M)\\ \dfrac{k_1k_2}{M\left(k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}\right)}\left(\dfrac{y}{M}\right)^{-k_2-1}, & (M\le y\le b)\end{cases}. \tag{D4}$$

The mean of the distribution with the bounded support $[0, b]$ is finite for all values of the shape parameters $k_1,\ k_2>0$ and is given by the formula

$$\mu_b=\int_0^b y\,f_b(y)dy=\frac{k_1k_2M}{(k_1+1)(k_2-1)}\cdot\frac{k_1+k_2-(k_1+1)\left(\frac{b}{M}\right)^{-k_2+1}}{k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}}=\mu\frac{k_1+k_2-(k_1+1)\left(\frac{b}{M}\right)^{-k_2+1}}{k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}}.$$

The median of the truncated double Pareto distribution equals $me_b=M\left(\sqrt[k_2]{\frac{2k_1}{k_1+k_2+k_1\left(\frac{b}{M}\right)^{-k_2}}}\right)$.

Therefore, for given mode M_b, median me_b and the mean μ_b of the truncated double Pareto distribution, its parameters k_1 and k_2 can be calculated using the formulas

$$\begin{cases}k_1=\dfrac{k_2\left(\frac{me_b}{M}\right)^{k_2}}{2-\left(\frac{me_b}{M}\right)^{k_2}-\left(\frac{me_b}{b}\right)^{k_2}};\\ \dfrac{\mu_b}{M}=\dfrac{k_1k_2}{(k_1+1)(k_2-1)}\cdot\dfrac{k_1+k_2-(k_1+1)\left(\frac{b}{M}\right)^{-k_2+1}}{k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}}.\end{cases}$$

Consider the Gini coefficient of the distribution $G_b=\frac{1}{\mu_b}\int_0^b F_b(y)\big(1-F_b(y)\big)dy$.

Denote $A=k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}$. Then the following relations hold:

$$\int_0^b F_b(y)\big(1-F_b(y)\big)dy=\frac{M}{A}\left[A_0+A_1+A_2+A_3-\frac{b}{M}\left(A_1+A_2\left(\frac{b}{M}\right)^{-k_2}+A_3\left(\frac{b}{M}\right)^{-2k_2}\right)\right],$$

where $A_0=\frac{k_2}{k_1+1}-\frac{k_2^2}{A(2k_1+1)}$, $A_1=\frac{(k_1+k_2)^2}{A}-k_1-k_2$, $A_2=\frac{2k_1(k_1+k_2)}{A(k_2-1)}-\frac{k_1}{k_2-1}$, $A_3=-\frac{k_1^2}{A(2k_2-1)}$.

Thus, $G_b=\frac{M}{\mu_bA}\left[A_0+A_1+A_2+A_3-\frac{b}{M}\left(A_1+A_2\left(\frac{b}{M}\right)^{-k_2}+A_3\left(\frac{b}{M}\right)^{-2k_2}\right)\right]$, or

$$G_b=\frac{(k_1+1)(k_2+1)}{k_1k_2\left[k_1+k_2-(k_1+1)\left(\frac{b}{M}\right)^{-k_2+1}\right]}\left[A_0+A_1+A_2+A_3-\frac{b}{M}\left(A_1+A_2\left(\frac{b}{M}\right)^{-k_2}+A_3\left(\frac{b}{M}\right)^{-2k_2}\right)\right].$$

Maximum likelihood estimation of the parameters k_1 and k_2 of the double Pareto distribution

Consider a sample $Y=\{y_1,\ldots, y_n\}$ of observations on household incomes. Suppose that the population distribution of the income variable I is double Pareto with the parameters k_1 and k_2 and the pdf $f(y|k_1,k_2)$ in (D1). The likelihood function for the sample Y is $\tilde{f}(y_1,\ldots y_n|k_1,k_2) = \prod_{i=1}^{n} f(y_i|k_1,k_2)$. The maximum likelihood estimates $\hat{k}_1$ and $\hat{k}_2$ of the parameters k_1 and k_2 are thus solutions to the problem

$$(\hat{k}_1,\ \hat{k}_2) = \arg\max_{k_1,k_2} \tilde{f}(x_1,\ldots x_n|k_1,k_2).$$

or, equivalently,

$$(\hat{k}_1,\ \hat{k}_2) = \arg\max_{k_1,k_2} \textstyle\sum_{i=1}^{n} \ln\,[f(y_i|k_1,k_2)]. \tag{D5}$$

Let n_1 be the number of observations in the sample such that $y_i<M$, $i=1,\ldots, n_1$. Then problem (D5) is reduced to maximization of the following function:

$$L(k_1,k_2) = \sum_{i=1}^{n_1} \ln\left[\frac{k_1k_2}{k_1+k_2}\frac{1}{M}\left(\frac{y}{M}\right)^{k_1-1}\right] + \sum_{i=n_1+1}^{n} \ln\left[\frac{k_1k_2}{k_1+k_2}\frac{1}{M}\left(\frac{y}{M}\right)^{-k_2-1}\right].$$

The function $L(k_1,k_2)$ can be transformed to

$$L(k_1,k_2) = Ak_1 + Bk_2 + n\cdot\ln\left(\frac{k_1k_2}{k_1+k_2}\right) - C,$$

where the constants A, B and C for the sample $Y=\{y_1,\ldots, y_n\}$ are defined as follows:

$$A = \sum_{i=1}^{n_1} \ln(y_i) - n_1\ln(M),\quad B = (n-n_1)\ln(M) - \sum_{i=n_1+1}^{n} \ln(y_i),\quad C = \sum_{i=1}^{n} \ln(y_i).$$

Consequently, problem (D5) is reduced to solving the system of equations $\begin{cases} \frac{\partial L(k_1,k_2)}{\partial k_1} = 0; \\ \frac{\partial L(k_1,k_2)}{\partial k_2} = 0, \end{cases}$ where $\frac{\partial L(k_1,k_2)}{\partial k_1} = A + \frac{nk_2}{k_1(k_1+k_2)}$ and $\frac{\partial L(k_1,k_2)}{\partial k_2} = B + \frac{nk_1}{k_2(k_1+k_2)}$. The solution to the system of equations $\begin{cases} A + \frac{nk_2}{k_1(k_1+k_2)} = 0; \\ B + \frac{nk_1}{k_2(k_1+k_2)} = 0, \end{cases}$ that maximizes the likelihood function $\tilde{f}(y_1,\ldots y_n|k_1,k_2)$ is given by the vector $(\hat{k}_1,\ \hat{k}_2) = \left(\frac{n}{\sqrt{AB}-A}, \frac{n}{\sqrt{AB}-B}\right)$.

Maximum likelihood estimation of the coefficients k_1 and k_2 of the truncated double Pareto distribution

Consider, as before, a sample $Y=\{y_1,\ldots, y_n\}$ of observations on household incomes. Suppose now that the population distribution of the income variable Y is truncated Pareto with the shape parameters k_1 and k_2, the truncation bound b and the pdf $f(y|k_1,k_2,b)$ in (D4).

It is easy to see that the maximum likelihood estimator $\hat{b}$ of the truncation parameter b is given by the maximal income level in the sample Y: $\hat{b} = \max_{1\le i\le n} y_i$. For $y_1,\ldots, y_n \le b$, the log-likelihood of the sample Y is given by

$$L(k_1,k_2)=\sum_{i=1}^{n_1}\ln\left[\frac{k_1k_2}{M\left(k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}\right)}\left(\frac{y_i}{M}\right)^{k_1-1}\right]+\sum_{i=n_1+1}^{n}\ln\left[\frac{k_1k_2}{M\left(k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}\right)}\left(\frac{y_i}{M}\right)^{-k_2-1}\right].$$

The log-likelihood function $L(k_1,k_2)$ can be transformed to

$$L(k_1,k_2)=n\cdot\ln(k_1k_2)-n\cdot\ln\left(k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}\right)+Ak_1+Bk_2-C,$$

where the constants A, B and C for the sample $Y=\{y_1,\dots,y_n\}$ are defined as follows:

$$A=\sum_{i=1}^{n_1}\ln(y_i)-n_1\ln(M),\quad B=(n-n_1)\ln(M)-\sum_{i=n_1+1}^{n}\ln(y_i),\quad C=\sum_{i=1}^{n}\ln(y_i).$$

The partial derivatives of the function $L(k_1,k_2)$ are given by $\frac{\partial L(k_1,k_2)}{\partial k_1}=A+\frac{nk_2}{k_1\left(k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}\right)}$, $\frac{\partial L(k_1,k_2)}{\partial k_2}=B+\frac{nk_1\left(1-\left(\frac{b}{M}\right)^{-k_2}-k_2^2\left(\frac{b}{M}\right)^{-k_2-1}\right)}{k_2\left(k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}\right)}$. Consequently, the maximum of the function $L(k_1,k_2)$ is achieved at the solutions of the following system of equations:

$$\begin{cases}\dfrac{nk_2}{k_1\left(k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}\right)}=-A;\\[2ex]\dfrac{nk_1\left(1-\left(\frac{b}{M}\right)^{-k_2}-k_2^2\left(\frac{b}{M}\right)^{-k_2-1}\right)}{k_2\left(k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}\right)}=-B.\end{cases}\tag{D5}$$

System of equations (D5) is equivalent to $\begin{cases}k_1^2=\dfrac{Bk_2^2}{A\left(1-\left(\frac{b}{M}\right)^{-k_2}-k_2^2\left(\frac{b}{M}\right)^{-k_2-1}\right)};\\[2ex]\dfrac{nk_1\left(1-\left(\frac{b}{M}\right)^{-k_2}-k_2^2\left(\frac{b}{M}\right)^{-k_2-1}\right)}{k_2\left(k_1+k_2-k_1\left(\frac{b}{M}\right)^{-k_2}\right)}=-B.\end{cases}$. The solution to the latter system of equation that maximizes the likelihood function is given by the vector $(\hat{k}_1,\ \hat{k}_2)$ such that

$$\hat{k}_1=\sqrt{\frac{B}{A}}\frac{\hat{k}_2}{\sqrt{1-\left(\frac{b}{M}\right)^{-\hat{k}_2}-\hat{k}_2^2\left(\frac{b}{M}\right)^{-\hat{k}_2-1}}},$$

and $\hat{k}_2$ is the solution of the following equation:

$$1-\left(\frac{b}{M}\right)^{-k_2}-k_2^2\left(\frac{b}{M}\right)^{-k_2-1}-\frac{k_2}{n}\left[\sqrt{AB\left(1-\left(\frac{b}{M}\right)^{-k_2}-\bar{k}_2^2\left(\frac{b}{M}\right)^{-k_2-1}\right)}-B+B\left(\frac{b}{M}\right)^{-k_2}\right]=0.$$